

Dynamic Analysis of a Ball-on-a-String:

Angular Momentum Conservation with Non-Ideal Torques

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1 Introduction and Problem Statement

We consider a ball-on-a-string being twirled horizontally by hand. The string is pulled inward, reducing the radius of rotation over a specific time interval. According to Euler's Second Law of rigid body dynamics, the rate of change of angular momentum $\mathbf{L}$ is equal to the sum of the external torques τ_{ext} applied to the system:

$$\frac{d\mathbf{L}}{dt} = \sum \tau_{\text{ext}} \quad (1)$$

The given parameters for the maneuver are:

- Initial rotational frequency, $f_i = 2 \text{ rev/s}$ (120 RPM, $\omega_i = 4\pi \text{ rad/s} \approx 12.57 \text{ rad/s}$)
- Initial radius, $r_i = 0.5 \text{ m}$
- Final radius, $r_f = 0.05 \text{ m}$ (10% of original)
- Time of transition, $\Delta t = 1.0 \text{ s}$

If there were zero external torques acting on the ball in the plane of rotation, angular momentum $L = mr^2\omega$ would be perfectly conserved. However, we must account for real-world decelerating torques: aerodynamic drag, mechanical friction at the hand pivot, and wobble.

2 Assumptions and System Modeling

To perform the numerical calculations, reasonable physical parameters must be estimated for a typical hand-held toy:

- **Mass of the ball (m):** 0.05 kg (50 grams)
- **Dimensions of the ball:** Modeled as a sphere with radius $R_b = 0.02 \text{ m}$, yielding a cross-sectional area $A = \pi R_b^2 \approx 0.001257 \text{ m}^2$.
- **Aerodynamics:** Standard sea-level air density $\rho = 1.225 \text{ kg/m}^3$ and a spherical drag coefficient $C_D = 0.47$.
- **Pivot Friction and Wobble:** To account for the observed prominent wobble in real hand-held systems, we model an effective pivot/wobble radius of $r_{\text{eff}} = 0.02 \text{ m}$ and an increased effective kinetic friction coefficient of $\mu_{\text{eff}} = 0.60$.
- **Radial Velocity (v_r):** We assume the string is pulled at a constant radial speed $v_r = \frac{r_i - r_f}{\Delta t} = 0.45 \text{ m/s}$. Thus, $r(t) = r_i - v_r t$.

3 Torque Contributions

The total external torque opposing the rotation consists of two primary components: τ_D (Drag) and τ_F (Friction/Wobble).

3.1 Aerodynamic Drag Torque (τ_D)

The drag force acting on the ball is $F_D = \frac{1}{2}\rho C_D A v^2$, where the tangential velocity is $v = r\omega$. The torque is $\tau_D = rF_D$:

$$\tau_D = \frac{1}{2}\rho C_D A r^3 \omega^2 \quad (2)$$

Expressing ω in terms of angular momentum L ($\omega = L/(mr^2)$):

$$\tau_D = \frac{\rho C_D A}{2m^2 r} L^2 = \frac{C_1}{r} L^2 \quad \text{where} \quad C_1 = \frac{\rho C_D A}{2m^2} \approx 0.1447 \text{ m}^{-1} \text{ kg}^{-1} \quad (3)$$

3.2 Friction and Wobble Torque (τ_F)

As the ball spins, the centripetal force generates high string tension $F_T = mr\omega^2$. When the string rotates against the edges of the hand or tube, and prominent wobble amplifies the effective contact area, it creates a frictional torque $\tau_F = \mu_{\text{eff}} r_{\text{eff}} F_T$:

$$\tau_F = \mu_{\text{eff}} r_{\text{eff}} \left(mr \frac{L^2}{m^2 r^4} \right) = \frac{\mu_{\text{eff}} r_{\text{eff}}}{mr^3} L^2 = \frac{C_2}{r^3} L^2 \quad (4)$$

Where $C_2 = \frac{\mu_{\text{eff}} r_{\text{eff}}}{m} = \frac{0.60 \times 0.02}{0.05} = 0.240 \text{ m} \cdot \text{kg}^{-1}$.

4 Integration of Euler's Equation

Euler's equation becomes a non-linear differential equation:

$$\frac{dL}{dt} = -\tau_{\text{total}} = - \left(\frac{C_1}{r} + \frac{C_2}{r^3} \right) L^2 \quad (5)$$

Using the chain rule, $\frac{dL}{dt} = \frac{dL}{dr} \frac{dr}{dt} = -v_r \frac{dL}{dr}$. Substituting this into the equation yields:

$$\frac{dL}{L^2} = \frac{1}{v_r} \left(\frac{C_1}{r} + \frac{C_2}{r^3} \right) dr \quad (6)$$

Integrating from the initial state (r_i, L_i) to the final state (r_f, L_f):

$$\int_{L_i}^{L_f} \frac{dL}{L^2} = \frac{1}{v_r} \int_{r_i}^{r_f} \left(\frac{C_1}{r} + \frac{C_2}{r^3} \right) dr \quad (7)$$

Evaluating the integrals:

$$\left[-\frac{1}{L} \right]_{L_i}^{L_f} = \frac{1}{v_r} \left[C_1 \ln(r) - \frac{C_2}{2r^2} \right]_{r_i}^{r_f} \quad (8)$$

$$\frac{1}{L_i} - \frac{1}{L_f} = \frac{1}{0.45} \left(C_1 \ln \left(\frac{0.05}{0.5} \right) - \frac{C_2}{2} \left(\frac{1}{0.05^2} - \frac{1}{0.5^2} \right) \right) \quad (9)$$

Substituting the constants $C_1 = 0.1447$ and $C_2 = 0.240$:

$$\frac{1}{L_i} - \frac{1}{L_f} = 2.222 (0.1447 \ln(0.1) - 0.12(400 - 4)) \quad (10)$$

$$= 2.222 (0.1447(-2.3026) - 0.12(396)) \quad (11)$$

$$= 2.222 (-0.3332 - 47.52) = -106.33 \text{ (kg} \cdot \text{m}^2/\text{s)}^{-1} \quad (12)$$

5 Calculations and Final Velocity

First, we find the initial angular momentum L_i :

$$L_i = mr_i^2 \omega_i = 0.05 \times (0.5)^2 \times (4\pi) \approx 0.1571 \text{ kg} \cdot \text{m}^2/\text{s} \quad (13)$$

Using our derived relation to find L_f :

$$\frac{1}{L_f} = \frac{1}{0.1571} + 106.33 = 6.365 + 106.33 = 112.695 \implies L_f = 0.00887 \text{ kg} \cdot \text{m}^2/\text{s} \quad (14)$$

Notice that angular momentum has dropped by over 94% due to the extreme friction, drag, and prominent wobble incurred as the velocity spikes. Finally, we convert L_f back to velocity:

$$\omega_f = \frac{L_f}{mr_f^2} = \frac{0.00887}{0.05 \times (0.05)^2} = \frac{0.00887}{0.000125} \approx 71.0 \text{ rad/s} \quad (15)$$

$$v_f = r_f \omega_f = 0.05 \times 71.0 \approx 3.55 \text{ m/s} \quad (16)$$

This predicted final angular velocity of 71.0 rad/s corresponds to roughly 11.3 rev/s, or 678 RPM.

6 Conclusion and Comparison

To fully appreciate the impact of the decelerating torques, we first calculate the theoretical ideal limit where external torques are zero and angular momentum is perfectly conserved ($L_f = L_i$). Under ideal conditions, the final angular velocity $\omega_{f,\text{ideal}}$ is derived purely from the ratio of the radii squared:

$$mr_i^2 \omega_i = mr_f^2 \omega_{f,\text{ideal}} \implies \omega_{f,\text{ideal}} = \omega_i \left(\frac{r_i}{r_f} \right)^2 \quad (17)$$

Substituting our initial values:

$$\omega_{f,\text{ideal}} = 4\pi \left(\frac{0.5}{0.05} \right)^2 = 4\pi(100) \approx 1256.6 \text{ rad/s} \quad (18)$$

Ideal System Results (Conservation of Angular Momentum):

- **Final Angular Velocity:** 1256.6 rad/s (200 rev/s, **12,000 RPM**)
- **Final Tangential Velocity:** 62.83 m/s

Predicted Real-World Results (With Drag, Friction, and Wobble):

- **Final Angular Velocity:** **71.0 rad/s** (approx. **11.3 rev/s**, **678 RPM**)
- **Final Tangential Velocity:** **3.55 m/s**

The presence of air resistance combined with prominent wobble and mechanical friction heavily bleeds angular momentum from the system. Because the frictional torque scales with $1/r^3$ and the string tension spikes drastically, these non-ideal effects restrict the real-world final velocity to well under 1000 RPM, a mere fraction ($\sim 5\%$) of the theoretical ideal limit.