

On the Conservation of Angular Momentum Under Varying Position and Momentum Vectors

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Abstract. A rare misconception in classical mechanics is the assertion that the angular momentum vector, defined as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, must necessarily change if the magnitudes and directions of the position vector $\mathbf{r}$ and linear momentum vector $\mathbf{p}$ are changing over time. This paper provides a rigorous mathematical proof demonstrating that $\mathbf{L}$ can remain strictly conserved despite continuous, simultaneous variations in the magnitudes of both $\mathbf{r}$ and $\mathbf{p}$. We show that the cross-product geometry naturally enforces dynamical conservation under central forces or free linear motion.

Keywords: angular momentum, central forces, cross product, areal velocity, classical mechanics, physics education

1 Introduction

In classical mechanics, the angular momentum $\mathbf{L}$ of a point mass about an origin is defined as the cross product of its position vector $\mathbf{r}$ and its linear momentum $\mathbf{p}$:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{1}$$

A rare misconception [1] asserts that if both $\mathbf{r}$ and $\mathbf{p}$ are changing with respect to time, the product $\mathbf{L}$ must also change, thereby invalidating the concept of conserved angular momentum in such dynamic systems. This assertion rests on a fundamental misunderstanding of vector calculus. In a vector cross product, the continuous variation of the constituent vectors does not mandate a variation in their product, provided their specific geometric relationship satisfies certain dynamic constraints.

2 Mathematical Proof of Conservation

To determine how angular momentum changes over time, we take the time derivative of $\mathbf{L}$. By applying the product rule for cross products, we obtain:

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \left(\frac{d\mathbf{r}}{dt} \times \mathbf{p} \right) + \left(\mathbf{r} \times \frac{d\mathbf{p}}{dt} \right) \tag{2}$$

Let us evaluate the two terms on the right side of Equation 2 independently.

First, recall the definition of velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ and linear momentum $\mathbf{p} = m\mathbf{v}$, where m is the constant mass of the particle. The first term becomes:

$$\frac{d\mathbf{r}}{dt} \times \mathbf{p} = \mathbf{v} \times (m\mathbf{v}) = m(\mathbf{v} \times \mathbf{v}) \quad (3)$$

Because the cross product of any vector with itself is exactly zero ($\mathbf{A} \times \mathbf{A} = 0$), the first term vanishes entirely: $m(\mathbf{v} \times \mathbf{v}) = 0$.

Second, from Newton's Second Law, the time derivative of linear momentum is the net force acting on the particle, $\frac{d\mathbf{p}}{dt} = \mathbf{F}$. Substituting this into the second term of Equation 2 yields:

$$\mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{r} \times \mathbf{F} = \boldsymbol{\tau} \quad (4)$$

where $\boldsymbol{\tau}$ is the net torque acting on the particle.

Therefore, the general time evolution of angular momentum simplifies to:

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} \quad (5)$$

Conclusion of the proof: If the net torque $\boldsymbol{\tau} = 0$, then $\frac{d\mathbf{L}}{dt} = 0$, which means $\mathbf{L}$ is a constant vector. This conservation holds entirely independently of whether $\mathbf{r}$ and $\mathbf{p}$ are changing, as long as the cross product of $\mathbf{r}$ and the applied force $\mathbf{F}$ is zero.

3 Algebraic and Geometric Perspective

The conservation of angular momentum can also be understood purely algebraically and geometrically by examining what the cross product inherently represents. Geometrically, the magnitude of a cross product $|\mathbf{r} \times \mathbf{p}|$ is equivalent to the area of the parallelogram formed by the vectors $\mathbf{r}$ and $\mathbf{p}$.

Consider the area A swept out by the position vector $\mathbf{r}$ as the particle moves through space. In a differential time interval dt , the particle undergoes a displacement $d\mathbf{r} = \mathbf{v}dt$. The differential triangular area dA swept out by the position vector is exactly half the area of the parallelogram formed by $\mathbf{r}$ and $d\mathbf{r}$:

$$dA = \frac{1}{2}|\mathbf{r} \times d\mathbf{r}| = \frac{1}{2}|\mathbf{r} \times \mathbf{v}|dt \quad (6)$$

Dividing by dt , the areal velocity (the rate at which area is swept out by the position vector) is:

$$\frac{dA}{dt} = \frac{1}{2}|\mathbf{r} \times \mathbf{v}| \quad (7)$$

Since linear momentum is $\mathbf{p} = m\mathbf{v}$, we can substitute $\mathbf{v} = \mathbf{p}/m$ into the areal velocity equation:

$$\frac{dA}{dt} = \frac{1}{2m}|\mathbf{r} \times \mathbf{p}| = \frac{|\mathbf{L}|}{2m} \quad (8)$$

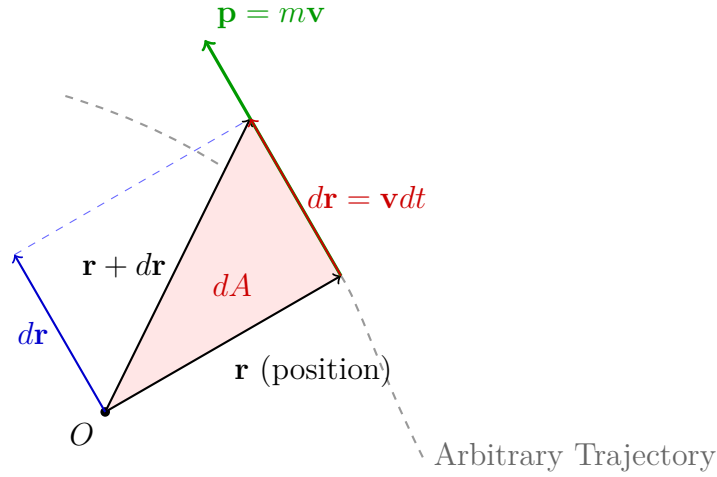


Figure 1. Geometric representation of the swept area dA . The area of the shaded triangle (dA) is exactly half the area of the parallelogram formed by $\mathbf{r}$ and $d\mathbf{r}$. Because the momentum vector $\mathbf{p} = m\mathbf{v}$ is collinear with the tangential displacement $d\mathbf{r}$, the magnitude of the cross product $\mathbf{r} \times \mathbf{p}$ scales directly with this swept area.

Algebraically, this demonstrates that the magnitude of the angular momentum, $|\mathbf{L}|$, is exactly $2m$ times the areal velocity $\frac{dA}{dt}$. Therefore, asserting that $\mathbf{L}$ must change if $\mathbf{r}$ and $\mathbf{p}$ change is mathematically equivalent to asserting that the swept area per unit time must change. Geometrically, as long as a particle moves such that it sweeps out equal areas in equal times (which is geometrically true for any central force or uniform linear motion), the areal velocity $\frac{dA}{dt}$ is constant. Thus, the algebraic value of $\mathbf{L}$ must remain perfectly conserved, irrespective of the fluctuating individual magnitudes and directions of $\mathbf{r}$ and $\mathbf{p}$.

4 Physical Examples

To make this mathematically abstract concept concrete, we consider two common physical scenarios where $\mathbf{r}$ and $\mathbf{p}$ change, yet $\mathbf{L}$ remains perfectly conserved.

4.1 Example 1: A Free Particle in Linear Motion

Consider a particle moving in a straight line at a constant velocity $\mathbf{v}$ that does not pass through the origin.

- The momentum $\mathbf{p} = m\mathbf{v}$ is constant.
- The position vector $\mathbf{r}$ is continuously changing in both magnitude and direction as the particle moves along its path.

Despite $\mathbf{r}$ continuously changing in both magnitude and direction, the magnitude of angular momentum is $L = |\mathbf{r}||\mathbf{p}|\sin(\theta)$, where θ is the angle between $\mathbf{r}$ and $\mathbf{p}$. Geometrically, the term $|\mathbf{r}|\sin(\theta)$ resolves exactly to the perpendicular distance (often called the impact parameter, b) from the origin to the linear path. Because this perpendicular distance b is rigorously constant

for a straight line, and the linear momentum $p = m|\mathbf{v}|$ is constant, their product $L = pb$ remains perfectly conserved.

4.2 Example 2: Planetary Orbits (Central Forces)

Consider a planet in an elliptical orbit around a star.

- The position vector $\mathbf{r}$ continuously changes in magnitude (distance to the star) and direction.
- The momentum vector $\mathbf{p}$ continuously changes in magnitude (orbital speed) and direction.

Gravity is a central force, meaning the force vector $\mathbf{F}$ is always directed strictly anti-parallel to the position vector $\mathbf{r}$. Therefore, the cross product $\mathbf{r} \times \mathbf{F} = \mathbf{0}$. By Equation 5, the torque is zero, and $\mathbf{L}$ is exactly conserved.

To understand how this conservation works dynamically despite the changing vectors, we can examine the scalar magnitude of the angular momentum, defined as $L = |\mathbf{r}||\mathbf{p}|\sin(\theta)$, where θ is the angle between the position and momentum vectors.

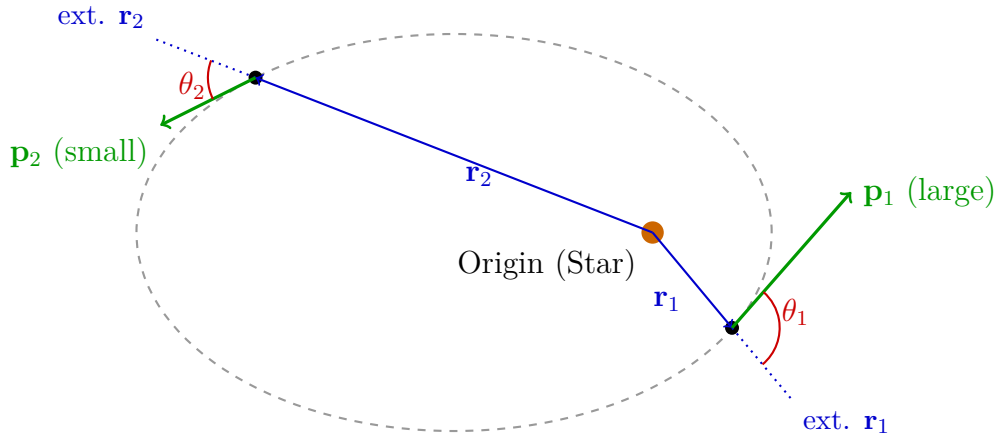


Figure 2. Dynamic balancing of variables in an elliptical orbit. At Position 1, the planet is close to the star ($|\mathbf{r}_1|$ is small), so it moves rapidly ($|\mathbf{p}_1|$ is large). At Position 2, the planet is far from the star ($|\mathbf{r}_2|$ is large), so it moves slowly ($|\mathbf{p}_2|$ is small). Furthermore, the angles θ_1 and θ_2 continuously change. The product $L = |\mathbf{r}||\mathbf{p}|\sin(\theta)$ perfectly balances these three fluctuating variables simultaneously to maintain a rigorously constant angular momentum.

As illustrated in Figure 2, the magnitude of the position vector ($|\mathbf{r}|$) varies as the planet moves between its closest approach and furthest distance. The magnitude of the momentum ($|\mathbf{p}|$) also varies, with the planet accelerating as it nears the star and decelerating as it moves away. Crucially, the angle θ between them is rarely 90° ; it changes continuously throughout the orbit. The geometric beauty of the cross product is that all three of these changing variables $|\mathbf{r}|$, $|\mathbf{p}|$, and $\sin(\theta)$ perfectly compensate for one another at every instantaneous point along the trajectory, yielding a single, constant value for L . This dynamic balancing act is precisely the geometric phenomenon famously codified as Kepler's Second Law (equal areas swept in equal times).

5 Summary

The assertion that $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ must change if $\mathbf{r}$ and $\mathbf{p}$ change is mathematically incorrect. The cross product is a geometric operation; as long as the dynamic changes in $\mathbf{r}$ and $\mathbf{p}$ are constrained such that no net torque is applied, the resultant vector $\mathbf{L}$ remains absolutely conserved.

References

- [1] In the angular momentum equation $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, which one of the remaining variables' magnitudes is correctly conserved when the magnitude of the radius changes? *ResearchGate Preprint* 321824496, https://www.researchgate.net/publication/321824496_In_the_angular_momentum_equation_L_r_x_p_which_one_of_the_remaining_variables'_magnitudes_is_correctly_conserved_when_the_magnitude_of_the_radius_changes